Outer Bound Frontier Analysis (“OBFA”).

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Abstract.

Farrell's (1957) envelopment of actual data points though the addition of points at infinity, on each axis, was given linear programming form as the CCR model Charnes et al. (1979). The CCR model frequently allocates zero prices to both inputs and outputs and much of Data Envelopment Analysis ("DEA") research has been towards elimination of zero prices or to the avoidance of their use in calculating efficiency. Varian (1984) suggested an alternative method to enveloping the actual data points, which tends to avoid the use of zero prices. A linear programming method of estimating Varian's suggested frontier is presented in this paper and given the name "Outer Bound Frontier Analysis" ("OBFA"). The method is shown to eliminate zero prices in a small example and to significantly reduce the incidence of zero prices in a series of Monte Carlo Experiments.

Section 1 – Introduction.

The history of DEA research from the first modification, in Charnes et al. (1979), which added the non-archimedian infinitesimal ε to the original model CCR model - Charnes et al. (1978) - has been a search for ways to restrict, modify or avoid the pricing consequences of Koopmans (1951) definition of efficiency and Farrell's (1957) method of drawing an envelopment around observed Decision Making Units ("DMUs").

Ali et al. (1995) give a useful framework for the classification of the various DEA models based upon three components of the models:

1. The pricing mechanism – bounds on prices,
2. The form of envelopment surface – constraints, and
3. The orientation – objective function.

The first component is an overt manipulation of prices: the other two components are implicit manipulations of the prices. Looking first at the bounds placed on prices. Dyson and Thanassoulis (1988) placed lower bounds on prices; Charnes et al. (1989) used a Cone Ratio approach; Beasley and Wong (1990) and Thompson et al. (1990) define Assurance Regions (“ARI” and “ARII”), using upper and lower bounds on prices that exclude vectors with unreasonable input and output prices. Cook et al. (1992) use lower bounds on prices to “break the tie” between technically efficient units and also present a model with weak ordinal weights constraints; seeking to force variables into the model in a particular order of importance. Doyle and Green (1994) and (1995) evaluate the efficiency of each DMU relative to the average efficiency of the each unit evaluated at the prices of all the other units. Green et al. (1996) develop an idea from Chang and Guh (1991) implementing price constraints by replacing the epsilon in the CCR model with “a data dependent finite magnitude”. Friedman and Sunuany-Stern (1997) use canonical correlation analysis to find prices to be used in models. Thanassoulis and Allen (1998) show the equivalence between the addition of a reduced set of “dummy” DMUs to the models and the assurance region constraint sets. Wei and Yu (1997) use K-cones to place restrictions on DEA solutions. For a good review of this topic see Allen et al. (1997).

Looking now at the second component of the models listed above: the form of envelopment surface. The use of priced inputs or priced outputs in the objective function with bounds on the prices alters the envelopment. In the Multiplicative Model, Charnes et al. (1983a) and (1983b) transform the observed data using logarithms and then derive an envelopment, which, when transformed back, is a continuous surface: thus altering both the envelopment and the prices. Andersen and Petersen's (1993) concept of super-efficiency, leaves each efficient unit, as it is being evaluated, out of the envelopment, such that it is compared to the convex combination of its nearest neighbors and returns a score of more than 100% efficient. Super-efficiency impacts both the envelopment and the prices. Bessent et al. (1988) present Constrained Facet Analysis ("CFA") in which the non fully dimensional facets are identified and adjustments are made for inefficient units. Lang et al. (1995) present Controlled Envelopment Analysis to address the issue that not all units will be either naturally or quasi-enveloped, as required by CFA. The Free Disposal Hull (FDH) model - Deprins et al. (1984) - envelopes the data using zero prices throughout. Thrall (1999) makes the point that FDH is not compatible with a real world economy where values (prices and costs) are considered important.

The third component of DEA models introduced above was the orientation or the form of the objective function. If the inputs (actual or related slacks) are the only side of the model included in the objective function then the model is input oriented. If the outputs (actual or related slacks) are the only side of the model included in the objective function then the model is output orientated. If both inputs and outputs are represented in the objective function then the model is non-oriented. The Additive Model from Charnes (1985) maximized the input and output slacks: efficiency scores are derived from a uniform pricing of the slacks. Green et al. (1997) modify this model with a non-linear objective.

Bardhan et al. (1996) presented a mixed integer model in which the comparison of inefficient units is with the actually observed efficient units rather than with convex combinations of units. Brockett et al. (1998) suggested a model in which the adjustment to the slack in the original Additive Model's objective function is not the absolute amount out the factor, but the range between the maximum and minimum observations of the factor. In other words they placed limits on the relationships between the prices of the slacks. Pastor et al. (1999) extended the previous Russell Measures of Efficiency with models that use prices of one for both inputs and outputs. Tone (2001) presents an oriented slack based model ("SBM").

One way or another these approaches seek to deal with the problem of zero prices. The original solution to the pricing problem was to add in the non-archimedian infinitesimal. This gave linear-programming form to Farrell's (1957) "points at infinity" on the axes that allowed the envelopment to be complete. Varian (1984) suggested an alternative approach to drawing the envelopment: instead of resting the planes of the envelopment on observed points with the points at the vertices, he proposed that planes should be balanced on the observed points. Since this method does not require the addition of points at infinity, it should not require price constraints to avoid zero prices.

This paper presents a constant returns to scale, oriented, three stage method - called the "Outer Bound Frontier Analysis ("OBFA") - that estimates an envelopment following Varian's suggested approach. The paper is organized as follows: Section 2 reviews Farrell's (1957) Radial Measure of Efficiency and the dual of the CCR model that implemented Farrell's Radial Measure. Section 3 looks at Varian's Outer Bound Requirement Set in the context of Farrell's Inner Bound Requirement Set. Section 4 looks at the modifications to the CCR model that implement Varian's suggestion. Section 5 presents some results from the application of the method and Section 6 concludes that although zero prices are not avoided, completely, the method results in significantly fewer zero prices than does the CCR model.

Notation.

To assist the reader, the notation used in this paper is presented first. Units of analysis, known as Decision Making Units (DMUs) will number n, and the subscript j will be used for units numbered from 1 to n.

A DMU will have s outputs indexed by r from 1 to s. Actual values of outputs will be represented by the letter y and weightings applied to outputs will be represented by the letter u. The sum of the weighted outputs will therefore be: $\sum_{r=1}^{s} \mu_{rj} y_{rj}$ for any given unit, j.

Similarly a DMU will have inputs indexed by i from 1 to m. Actual values of inputs will be represented by the letter x and weightings applied to the inputs will be represented by the letter v. The sum of the weighted inputs will therefore be: $\sum_{i=1}^{m} v_{ij} x_{ij}$ for any given unit, j.

In DEA, a math model is run for every unit, so when the variable or price is specifically the variable or price for the unit under consideration, the subscript "0" replaces "j".

Output slack is represented by s_{ij}^{+} and input slack is represented by s_{ij}^{-}.

Where the representation of the model in the paper from which it was taken did not follow these conventions: it has been edited to conform, before being presented.

Section 2 - Farrell's Radial Measure of Efficiency and the CCR Model.

Koopmans (1951) modeled an entire economy as a convex polyhedral cone. He defined an efficient point and a frontier derived from repeated use of this definition is referred to as Pareto-Koopmans efficiency in DEA literature. Koopmans gives an economic interpretation of these efficiency conditions in the following theorem:

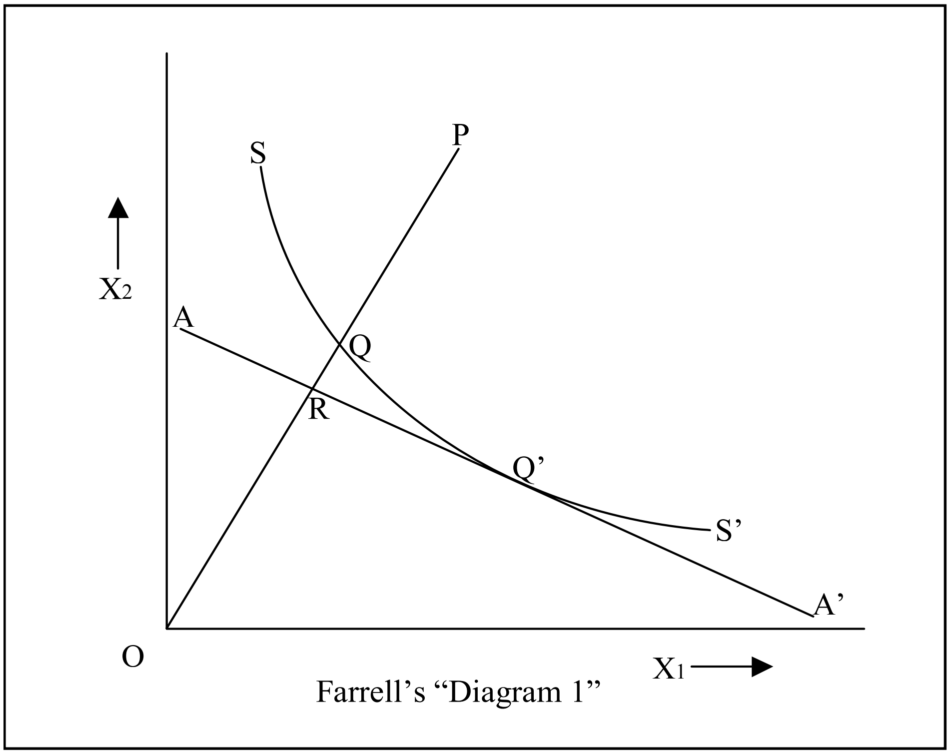
> THEOREM 4.7: A necessary and sufficient condition that the activity vector x shall lead to an efficient point y = Ax in the commodity space is that there exists a vector p of positive prices such that no activity in the technology permits a positive profit and such that the profit on all activities carried out at a positive level be zero."

At any particular point, j, on the frontier, s outputs y_{rj} (for r = 1 to s), are derived from the use of m inputs x_{ij} (for i = 1 to m). If the vector p of prices is split between outputs prices μ_r and inputs prices ν_i then the theorem translates into:

$\text{Profit}_j = 0 = \sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} \nu_i x_{ij}$ for a unit, j, on the efficient frontier.

Debreu (1951) and Farrell (1957) were concerned with measurement of the difference between efficient and inefficient points in a production possibilities set estimated by reference to actual observations. "Diagram 1." from Farrell is reproduced as Figure 1. The axes representing the quantities of the primary factors have been labeled x_1 and x_2 to be consistent with the notation used in this paper.

Figure 1 – Farrell's "Diagram 1."



SS' is an isoquant representing combinations of the primary factors x_1 and x_2 required to produce a single output at a given level (q) at technical efficiency.

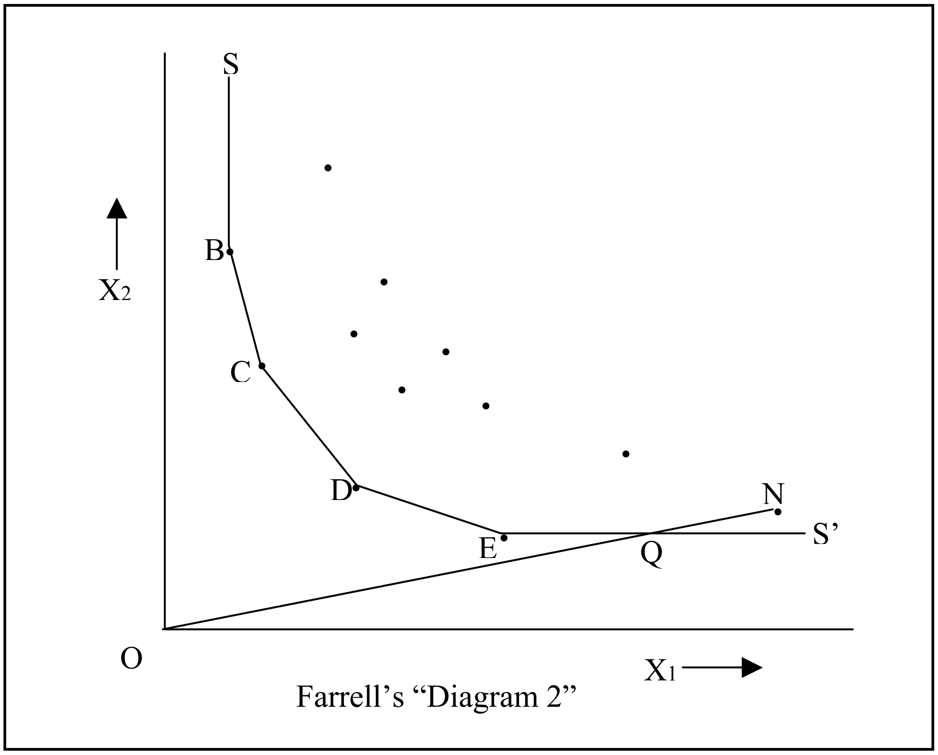
AA' is a market price line whose slope is the ratio of the market prices of the primary factors. Production of q of the output at Q' (where AA' is tangent to SS') is both technically efficient and price efficient. Q' represents the combination of the primary factors which results in the production of (q) at the lowest cost.

A unit operating at P can improve technical efficiency by reducing its usage of either or both of the primary factors. Assuming proportionate reductions in each primary factor, the unit would operate technically efficiently at Q. However to be as cost efficient as Q' at the market prices implied by AA' would require the (P) to be operating at R if it were possible to proportionately reduce its use of the primary factors.

Farrell takes the ratio OR/OQ as the measure of Price Efficiency of P and OQ/OP as the measure of Technical Efficiency of P. The Overall Efficiency of P is the Price Efficiency Technical Efficiency = OR/OQ OQ/OP = OR/OP.

Estimation of the Isoquant SS' from empirical data required a piecewise linear Isoquant to be drawn around actual points scaled to the output of one. Farrell's "Diagram 2", reproduced as Figure 2, shows an isoquant drawn around actual observations of the use of two inputs x_1 and x_2 to produce one unit of output by units B, C, D and E. It comprises the minimum set of convex combinations of the production possibilities or to put it another way straight lines between the observed efficient units.

Figure 2 – Farrell's "Diagram 2."



Without the addition of the points at infinity, S and S', a point like N would have no measurable radial efficiency as there would be no point Q intersecting the isoquant and the radial from O to N and therefore no measurement OQ/ON of N's technical efficiency.

If unit N were able to scale back, proportionately, its use of the two factors it would achieve "technical" efficiency at Q where its usage of x_2 would be the same as that of E, but its usage of x_1 would be greater that that of E. Nevertheless, Q is the yardstick by which Farrell would be measuring N's technical efficiency. But, Q is only 100% technically efficient because the extra x_1 used at Q over E attracts a zero price in terms of x_2. Q is in effect paying nothing for the extra x_1 that it uses when compared to E. This zero "price" results from the requirement to use points at infinity. If ES' were at an angle to the x_1 axis rather than normal to it then zero prices would not arise.

Charnes et al. (1978) formulated an approach to the solution to Farrell's measurement of efficiency for each of a number of observed DMUs. For each DMU, 0, they solved a Linear Program whose dual is:

$$\min g_0 = \sum_{i=1}^{m} v_{io} x_{io}$$

$$s.t.$$

$$(1) \qquad -\sum_{r=1}^{s} \mu_{ro} y_{rj} + \sum_{i=1}^{m} v_{io} x_{ij} + \sum_{i=1}^{m} s_{ij}^{-} = 0 \ \forall j$$

$$(2) \qquad \sum_{r=1}^{s} \mu_{ro} y_{rj} = 1 \ \forall j$$

$$(3) \qquad \mu_{ro}, v_{io} \geq 0 \ \ \forall \text{i and r}$$

$$(4) \qquad y_{rj}, x_{ij}, s_{ij}^{-} \geq 0 \ \forall \text{i, r and j}$$

Charnes et al. (1979) added the non-archimedian infinitesimal ε to the model, restricting the prices in the dual to being greater than ε as follows:

$\mu_r \geq \varepsilon \geq 0,\ \nu_i \geq \varepsilon \geq 0$ for all r, i .

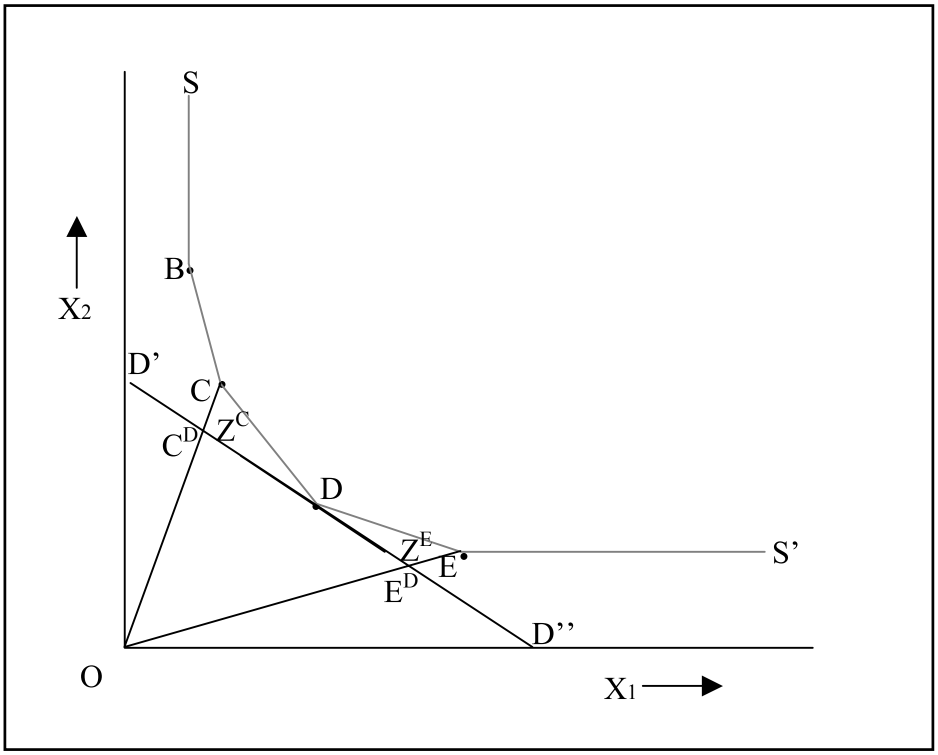
The use of ε results in a ranking of units derived from the selection of an arbitrary value. This has made use of ε controversial - Charnes and Cooper (1984). Selecting the value for ε is also a practical problem when it comes to the simplex. For more on this topic see Ali and Seiford (1993) and Mehrabian et al. (2000).

Section 3 - Varian's Outer Bound Requirement Set.

Of course, there are many different ways in which we can rest segments on points B, C, D and E in Farrell's Diagram 2. Think of a segment resting on D that splits the angular "distance" between the segment CD and DE – see Figure 3.

Varian's [1984] approach assumes that the segment rests on the point in such a way that the point is the center of the segment; which, in practice, would be a slightly different point to that suggested here. Varian [1984] describes the frontier used by Farrell as the "Inner" bound requirement set and suggested an "Outer" bound requirement set.

Figure 3 – The Addition of an “Outer” Bound Facet at D.

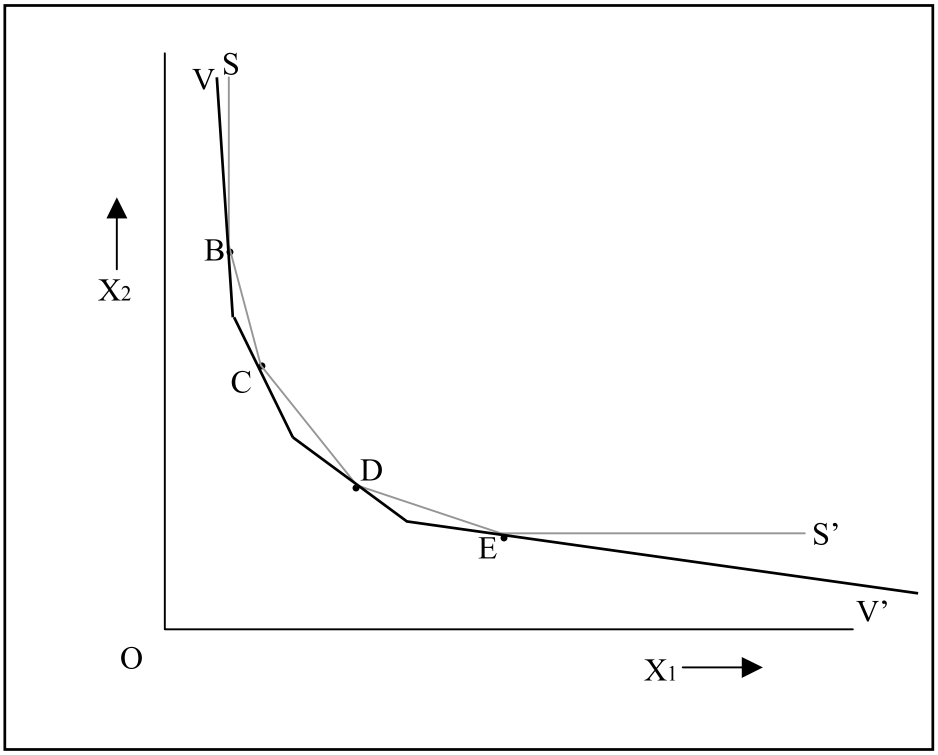


If D’D’’ is D’s price line, and C is evaluated with reference to D’D’’, then OC^D/OC is the “price” efficiency of C. If D’D’’ is D’s price line, and E is evaluated with reference to D’D’’, then OE^D/OE is the “price” efficiency of E.

If D is constrained to be price efficient at its own prices and both C and E are constrained to be price inefficient at D’s prices then the result will be to balance a segment similar to D’D’’ on D. Maximizing the inefficiency of B, C, and E at D’s prices and at the same time forcing D to be efficient at its own prices will achieve this result.

If we rest similar segments on points B, C and E, the new Isoquant would look as follows. Varian [1984] calls a frontier made up of such Isoquants the “Outer” bound requirement set – see Figure 4.

Figure 4 – An “Outer” Bound Requirement Set and An “Inner” Bound Requirement Set.



A problem arises in that there is no point above and to the left of B and no point below and to the right of E. Maximizing all other point’s price inefficiencies with respect to B’s prices will result in B adopting an infinite price for a unit of x_1 and a zero price for a unit of x_2. In other words the segment resting on B will be parallel to the x_2 axis. Similarly the segment resting on E will be parallel to the x_1 axis. So this model, like the CCR model, may require some countervailing action towards non-zero or non-infinite prices.

Section 4 - An Outer Bound Frontier Analysis Method

This method involves three steps.

1. The first step is to identify the Pareto-Koopmans efficient set of DMUs using the CCR DEA Model: set {Q}. The CCR model is run using the desired orientation and a set of efficient units is derived.

2. The second step is to place an "Outer" bound frontier can be placed around the efficient set of units identified in the first step.

3. The third step is to calculate the efficiency scores of the inefficient units by reference to the "Outer" bound frontier. This is simply a matter of calculating efficiency scores for each inefficient unit using the sets of prices derived from the second step and selecting the highest score for each inefficient unit.

The second step will be the focus of the remainder of this section. The method adopted is a modification to the Dual of the CCR DEA Model presented in Section 3. In estimating the "Outer" bound frontier, only the Q DMUs that are found to be efficient using the CCR Model are of interest. Let an efficient unit be denoted $k \in \{Q\}$ where {Q} is the set of efficient units, we can rewrite the constraints in the Dual CCR Model for a single DMU, k, as:

(1a) $$-\sum_{r=1}^{s} \mu_{rk} y_{rk} + \sum_{i=1}^{m} v_{ik} x_{ik} = 0$$

(2a) $$\sum_{r=1}^{s} \mu_{rk} y_{rk} = 1$$

(3a) $$\mu_{rk}, v_{ik}, y_{rk}, x_{ik} \geq 0 \ \forall i, r$$

(1a) holds because for an efficient unit the ratio $\sum_{i=1}^{m} v_{io} x_{io} \Big/ \sum_{r=1}^{s} \mu_{ro} y_{ro} = 1$. (2a) and (3a) now refer to a single unit, k. For another efficient unit, $k' \in \{Q\}$ and $k' \neq k$ to be inefficient when weighted or priced at μ_{rk} and v_{ik} requires that for k':

$$(4) \qquad -\sum_{r=1}^{s} \mu_{rk} y_{rk'} + \sum_{i=1}^{m} v_{ik} x_{ik'} \geq 0 \ \forall k'$$

Making constraint (4) into an equality by introducing a variable $d_{kk'}$, representing a distance from efficiency for units k' at the prices of k, then for k to be the only unit efficient at its own prices requires that:

$$(1a) \qquad -\sum_{r=1}^{s} \mu_{rk} y_{rk} + \sum_{i=1}^{m} v_{ik} x_{ik} = 0$$

$$(4a) \qquad -\sum_{r=1}^{s} \mu_{rk} y_{rk'} + \sum_{i=1}^{m} v_{ik} x_{ik'} + d_{kk'} = 0 \ \forall k'$$

$$(5) \qquad d_{kk'} \geq 0 \ \forall k'$$

Letting the objective function be $\max \sum_{k'} d_{kk'}$ would give the following linear program:

$$\max \sum_{k'} d_{kk'}$$

$$s.t.$$

$$(1a) \qquad -\sum_{r=1}^{s} \mu_{rk} y_{rk} + \sum_{i=1}^{m} v_{ik} x_{ik} = 0$$

$$(4a) \qquad -\sum_{r=1}^{s} \mu_{rk} y_{rk'} + \sum_{i=1}^{m} v_{ik} x_{ik'} + d_{kk'} = 0 \ \forall k'$$

$$(5) \qquad d_{kk'} \geq 0 \ \forall k'$$

$$(3a) \qquad \mu_{rk}, v_{ik}, y_{rk}, x_{ik} \geq 0 \ \forall i, r$$

Note that the scaling constraint (2a) is no longer required and has been dropped from the model.

Since maximizing the sum of the distances $d_{kk'}$ may result in some zero distances and this is not what is required we need to force all $d_{kk'}$ to be positive by adding a minimum distance variable, D_k, to the model; by amending constraint (5) to:

(5*a*) $d_{kk'} \geq D_k \ \forall k'$ and changing the objective function to: max D_k. The full model then looks as follows:

$\max D_k$

$s.t.$

(1a) $-\sum_{r=1}^{s} \mu_{rk} y_{rk} + \sum_{i=1}^{m} v_{ik} x_{ik} = 0$

(4a) $-\sum_{r=1}^{s} \mu_{rk} y_{rk'} + \sum_{i=1}^{m} v_{ik} x_{ik'} + d_{kk'} = 0 \ \forall k'$

(5*a*) $d_{kk'} \geq D_k \ \forall k'$

(3c) $\mu_{rk}, v_{ik}, y_{rk}, x_{ik}, y_{rk'}, x_{ik'}, D_k \geq 0 \ \forall i, r, k'$

This model maximizes the minimum distance, but at the cost of minimizing the prices. To maximize the minimum price: let α_k be the minimum input price for DMU k and β_k be the minimum output price for DMU k and add in constraints in the form:

(6) $\mu_{rk} - \beta_k \geq 0 \ \forall r$ and (7) $v_{ik} - \alpha_k \geq 0 \ \forall i$ and add $\max \beta_k + \alpha_k$ to the objective function and the following model results:

$$\max \beta_k + \alpha_k + \sum_{k \in \{Q\}} D_k$$

$s.t.$

(1c) $-\sum_{r=1}^{s} \mu_{rk} y_{rk} + \sum_{i=1}^{m} v_{ik} x_{ik} = 0 \ \forall k$

(4b) $-\sum_{r=1}^{s} \mu_{rk} y_{rk'} + \sum_{i=1}^{m} v_{ik} x_{ik'} + d_{kk'} = 0 \ \forall k, \forall k' \neq k$

$(5b)$ $d_{kk'} \geq D_k \ \forall k, k' \neq k$

(6) $\mu_{rk} - \beta_k \geq 0 \ \forall r$

(7) $v_{ik} - \alpha_k \geq 0 \ \forall i$

(3d) $y_{rk}, x_{ik}, y_{rk'}, x_{ik'}, D_k \geq 0 \ \forall i, r, k, k'$

$k \in \{Q\}, k' \in \{Q\}$

Solving this model k times for every unit k in the efficient set {Q} would give k sets of prices which in turn would define k facets of an “Outer Bound Frontier”. Unlike the DEA Model, which requires the solution of a Linear Program for each DMU, the Distance Model can be expanded to solve for prices for each efficient DMU in one LP, as follows:

$$\max \sum_{k \in \{Q\}} \alpha_k + \sum_{k \in \{Q\}} \beta_k + \sum_{k \in \{Q\}} D_k$$

$s.t.$

(1c) $-\sum_{r=1}^{s} \mu_{rk} y_{rk} + \sum_{i=1}^{m} v_{ik} x_{ik} = 0 \ \forall k$

(4b) $-\sum_{r=1}^{s} \mu_{rk} y_{rk'} + \sum_{i=1}^{m} v_{ik} x_{ik'} + d_{kk'} = 0 \ \forall k, \forall k' \neq k$

$(5b)$ $d_{kk'} \geq D_k \ \forall k, k'$

$(6a)$ $\mu_{rk} - \beta_k \geq 0 \ \forall r, k$

$(7a)$ $v_{ik} - \alpha_k \geq 0 \ \forall i, k$

(3f) $\alpha_k, \beta_k, y_{rk}, x_{ik}, y_{rk'}, x_{ik'}, D_k \geq 0$ ∀i, r, k, k'

$k \in \{Q\}, k' \in \{Q\}$

Input Orientation requires only that constraint (4b) be changed by changing the signs throughout, thus:

$$\max \sum_{k \in \{Q\}} \alpha_k + \sum_{k \in \{Q\}} \beta_k + \sum_{k \in \{Q\}} D_k$$

$$s.t.$$

$$\begin{array}{ll}
(1\text{c}) & -\sum_{r=1}^{s} \mu_{rk} y_{rk} + \sum_{i=1}^{m} v_{ik} x_{ik} = 0 \ \forall k \\
(4\text{c}) & \sum_{r=1}^{s} \mu_{rk} y_{rk'} - \sum_{i=1}^{m} v_{ik} x_{ik'} + d_{kk'} = 0 \ \forall k, \forall k' \neq k \\
(5b) & d_{kk'} \geq D_k \ \forall k, k' \\
(6a) & \mu_{rk} - \beta_k \geq 0 \ \forall r, k \\
(7a) & v_{ik} - \alpha_k \geq 0 \ \forall i, k \\
(3f) & \alpha_k, \beta_k, y_{rk}, x_{ik}, y_{rk'}, x_{ik'}, D_k \geq 0 \ \forall \text{i, r, k, k'} \\
 & k \in \{Q\}, k' \in \{Q\}
\end{array}$$

The appendix shows an implementation of this model – step two in the method – in MPL/CPLEX and the Access Database.

Section 5 – Results – Application of the Method.

The method was applied to the data for a simple one-output, two-input example – see Table 1.

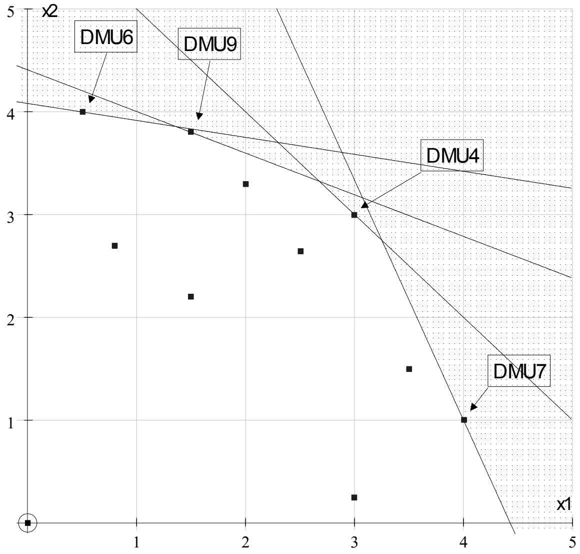
Table 1 - Example Data.

	Input	Outputs	
DMU	X′	Y1′	Y2′
1	3.500	12.250	5.250
2	3.000	2.400	8.100
3	1.800	2.700	3.960
4	1.600	4.800	4.800
5	1.100	2.750	2.915
6	1.500	0.750	6.000
7	2.600	10.400	2.600
8	2.500	7.500	0.625
9	3.500	5.250	13.300
10	1.700	3.400	5.610
11	3.200	9.600	9.600

First an output oriented CCR model was run to identify the efficient set of DMUs: Q={4,6,7,9,11}. The efficient units were compared and DMU 11 was found to be a scaling by a factor of two of DMU 4. Since the inclusion of two DMUs that are the same as each other, or scaled versions of one another, would force the maximum minimum distance to zero: DMU 11 was removed from the efficient set used in the OBFA model. This makes no difference to the result since, in step three DMU 11 will be evaluated at the prices for DMU 4 and will be 100 percent efficient at those prices.

The Outer Bound Frontier Analysis model was then run using the data for DMUs 4, 6, 7 and 9. The ratios of the prices for the inputs for the four efficient DMUs allowed an outer bound frontier to be derived and graphed around the ten DMUs, each scaled to outputs of one, i.e. plotted as an isoquant. Figure 5 shows the results: Note that there are no zero or infinite prices: no facet is normal to the axes. No facet connects two points and the minimum distance exists between DMU9 and the facet balanced on DMU6.

Figure 5 – Frontier from the “Both” Method For Two-Output One-Input Example.



Under ideal conditions, the method provides a good estimate of an outer bound frontier and removes zero prices from the results. Unfortunately the results still depend to some extent on the DMUs to either side of the DMU under consideration. The facet resting on DMU 7 in this example might have been a vertical line since only the inclusion of a maximum minimum price in the objective has made the facet recline. If the maximum minimum distance in the objective had been “stronger” than the maximum minimum price then the facet would have been a vertical line. The absence of a DMU, further to the right and lower than DMU 7 can still lead to a zero price which implies a vertical facet in this case. The facet at DMU6 might have been a horizontal line under similar conditions.

A series of Monte Carlo experiments were undertaken to establish whether the Outer Bound Frontier Analysis ("OBFA") method resulted in significantly fewer zero prices than the CCR model. The 40 models run had the following characteristics:

- 8 Model Sizes: 2 by 2, 2 by 3, 2 by 4, 2 by 6, 6 by 6, 6 by 2, 4 by 2 and 4 by 3 (# of inputs by # of outputs).
- 5 Sets Of Random Data.

This series of experiments confirmed – see Table 2 – that the Outer Bound Frontier Analysis method does result in significantly fewer zero prices than the CCR model.

Table 2 - Total number of zero prices for efficient units, by method. (out of 14,094 prices).

Method	CCR Model	"OBFA" Method
Number and Percentage	3,119 (22%)	1,417 (10%)

Section 6 – Conclusions.

The outer bound frontier analysis method and model provide a means to estimating a frontier drawn in accordance with Varian's suggestion. In the process it provides a means to obtaining radial efficiency results that are significantly less dependent on zero prices.

Appendix – An MPL/CPLEX/Access Implementation of the "OBFA" Method.

MPL can use an Access database as the source of the data-values and as a store for the results values. In anticipation of running a large number (thousands) of similar sets of data through the models and because calculation of the efficiency scores requires a "Cartesian" join; using an SQL database seemed to be, and proved to be, the natural alternative to the other options MPL provides for data input and data output - Excel and Text. The data inputs to the model are:

1. The Efficient Set of DMUs. These were stored as queries in Access represented in the model below as "IndexDMUQuery".

2. Names of the Outputs and of the Inputs Variables. These were stored as tables and are represented in the model as "IndexOutputsTable" and "IndexInputsTable" respectively. These three indices comprise vectors expressed in Access as single field tables or queries for which the column name was "field".

3. Input and Output Values for the DMUs. These were set up as queries in Access because MPL requires that only the values corresponding to the MPL indices for the model be in the model inputs in Access. The n*m matrix of input data and the n*s matrix of output data have to be defined in Access as separate tables or queries. The table or query has to be in the form of three columns; one for row index values (DMU) another for column index values (inputs index or outputs index), and the third column for the data.

The data outputs from the model are Prices and Maximum Minimum Distances. A similar design to that required for the data values input tables or queries, is required for the tables defined for the output from MPL. Tables were defined for each as "PricesTable" and "DistanceTable", and filled with the DMU and other Indices values, but with blanks for the data-values.

When writing data to Access, MPL ignores index values, in the Access Tables, that are not in the MPL indices. In addition no warning is given if the Access Table does not contain a full set of the rows needed if MPL is to write out all of the results values it calculates. This means that the tables are best defined with all possible and conceivable index values included. Additionally, it meant that the input and output prices could be captured in a single Table in Access. The MPL code for the "Both" method is given in Figure 6.

Figure 6 – The MPL Code for the "Both" Method.

```
{ModelName.mpl}

TITLE
      ModelTitle ;

OPTIONS
      DatabaseType=Access ;
      DatabaseAccess="P:\ACCESS.mdb";

INDEX
      Output := DATABASE("IndexOutputsTable","field");
      Input := DATABASE("IndexInputsTable","field");
      DMU := DATABASE("DMUIndexQuery","DMU"):4 CIRCULAR;
      DMU2 := DMU CIRCULAR;


DATA
      Outputs[DMU,Output]:= DATABASE("OutputsQuery","Outputs_Field",
            DMU="DMU_Field",Output="Index_Field");
      Inputs[DMU,Input]:= DATABASE("InputsQuery","Inputs_Field",
            DMU="DMU_Field",Input="Index_Field");
      ScalingConstant   := 1000000;
```

Figure 6 – The MPL Code for the “Both” Method cont.

```
DECISION VARIABLES
      OutputPrice[DMU,Output]       ->    OP
      EXPORT TO DATABASE("PricesTable","Price_Field",
            DMU="DMU",Output="Index_Field ");
      InputPrice[DMU,Input]         ->    IP
      EXPORT TO DATABASE("PricesTable","Price_Field",
            DMU="DMU",Input="Index_Field");
      EXPORT TO DATABASE("DistanceTable","Distance_Field",
            DMU="DMU");
      Distance[DMU,DMU]       ->    U;
      MinInpPrice[DMU]        ->    L;
      MinOutPrice[DMU]        ->    M;
      MinDistance[DMU]        ->    N

MACRO
      TotalMinDistance  :=SUM(DMU:MinDistance);
      TotalMinPrice     :=SUM(DMU:MinInpPrice) + SUM(DMU:MinOutPrice) ;

MAX Outcome = TotalMinDistance + TotalMinPrice;

SUBJECT TO
      Scaling[DMU]      -> Sca:
            SUM(Input:Inputs[DMU,Input]*InputPrice[DMU,Input]) -
            ScalingConstant = 0;

      Decidc[DMU,DMU2] WHERE (DMU=DMU2)   -> Dcc:
            SUM(Output:Outputs[DMU,Output]*OutputPrice[DMU:=DMU2,Output])
            - SUM(Input:Inputs[DMU,Input]*InputPrice[DMU:=DMU2,Input]) =
            0;

      Decide[DMU,DMU2] WHERE (DMU<>DMU2)  -> Dcd:
            SUM(Output:Outputs[DMU,Output]*OutputPrice[DMU:=DMU2,Output])
            - SUM(Input:Inputs[DMU,Input]*InputPrice[DMU:=DMU2,Input])
            + Distance[DMU:=DMU2,DMU]= 0;

      InputPrices[DMU,Input]  -> InP:
            InputPrice[DMU,Input] - MinInpPrice[DMU] > 0;

      OutputPrices[DMU,Output]      -> OuP:
            OutputPrice[DMU,Output] - MinOutPrice[DMU] > 0;

      Ds[DMU,DMU2] WHERE (DMU<>DMU2) :
            Distance[DMU:=DMU2,DMU] - MinDistance[DMU] > 0 ;

      Enforce[DMU] -> En:
            MinDistance[DMU] = MinDistance[DMU+1];

BOUNDS
      MinDistance[DMU] > 0;
      MinInpPrice[DMU] > 0;
      MinOutPrice[DMU] > 0;

END
```

Two other features of the implementation deserve comment. The first feature is that MPL performs an index type checking to ensure consistency in the definition of matrix variables and constraints. The actual index defined for a matrix or vector variable (or for a constraint), must be referred to when referencing the matrix or vector (or in the constraint definition). So, for example, the outputs matrix “Outputs[DMU,Output]” requires the indices “DMU” and “Output”.

If, as in the case of this model, you want to traverse a matrix whose rows are DMUs and whose columns are DMUs, you need to define a second DMU index. When you use the second DMU index, “DMU2” in this model, you must tell MPL that you are aware of the “index type conflict” by a process analogous to a “C” programming “type-cast” in the form “DMU:=DMU2”.

A more widespread use of the method will, like DEA did, require the development of a front end to the solver.

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